Problem 1 (30 points)

This problem mimics Exercise 1.1 from the CF textbook. Consider an overlapping generations economy with a constant population of \( N = 100 \) at every period \( t = 1, 2, 3, \ldots \) Regular individuals that live for two periods are endowed with \( y = 20 \) units of consumption good when young. They have no endowment when old. As usual, there is an initial old population in period \( t = 1 \).

Do the following:

(a) Write down the equation for the set of feasible allocations for this economy at every period \( t \).

(b) Suppose that every member of the initial old generation get \( m > 0 \) units of fiat money. Write down the per-period budget constraints for a regular individual born in period \( t \). Derive the lifetime budget constraint for this individual.

(c) Suppose that \( m = 4 \). Write down the money-market clearing condition for every period \( t \). Use this condition to find the real rate of return on fiat money in a stationary equilibrium.

For the rest of the problem suppose that

\[ v_t m_t = \frac{y}{1 + \frac{v_t}{v_{t+1}}} \]

(d) Compute the value of \( v_t \) for an arbitrary period \( t \).

(e) Suppose that every member of the initial old generation has utility function \( u(c_{0,1}) = \sqrt{c_{0,1}} \). Calculate their utility in the monetary equilibrium.

(f) Finally, suppose that \( m = 8 \), i.e. the initial money stock in this economy is double the original size. Recalculate the value of \( v_t \) and compare it with the one obtained in part (d). Are the initial old better-off from having more money as compared to the situation from part (e)?

Solution

(a) (5 points) The feasible set is

\[ N_1 c_{1,t} + N_{t-1} c_{2,t} \leq N_t y, \]

and this can be simplified using the fact that \( y = 20 \) and \( N \) is constant, to

\[ c_{1,t} + c_{2,t} \leq 20. \]
(b) (5 points) The first and second-period constraints are:

\[
\begin{cases}
    c_{1,t} + v_t m_t = 20 \\
    c_{2,t+1} = v_{t+1} m_t
\end{cases}
\]

To get the lifetime budget constraint, I eliminate \( m_t \), substituting it from the second equation into the first one, and obtain

\[
c_{1,t} + \frac{v_t}{v_{t+1}} c_{2,t+1} = 20.
\]

(c) (5 points) The money market clearing condition is:

\[N_t m = N_t m_t,
\]

and I write it in terms of consumption good as (canceling \( N_t \)):

\[v_t 4 = v_t m_t,
\]

and use the first period budget constraint to get:

\[v_t 4 = 20 - c_{1,t}
\]

\[v_t = \frac{20 - c_{1,t}}{4}.
\]

In a stationary equilibrium, \( c_{1,t} = c_1 \) for all \( t = 1, 2, 3, \ldots \) So I write:

\[\frac{v_{t+1}}{v_t} = \frac{20 - c_{1,t+1}}{4} = \frac{20 - c_1}{4} = 1.
\]

(d) (5 points) Since

\[v_t m_t = \frac{y}{1 + \frac{v_t}{v_{t+1}}},
\]

we know that

\[v_t m_t = \frac{20}{1 + 1} = 10.
\]

The money market clearing condition says

\[N_t m = N_t m_t,
\]

or just

\[m = m_t,
\]

and since \( m = 4 \), we can arrive at

\[v_t = \frac{v_t m_t}{m_t} = \frac{10}{4} = 2.5 \quad \text{for every} \ t = 1, 2, 3 \ldots,
\]

i.e. 1 unit of fiat money can purchase 2.5 units of consumption good.
(e) **(5 points)** The problem of an initial old person is:

$$\max_{c_{0,1}} \sqrt{c_{01}} \quad \text{s.t.} \quad c_{0,1} \leq v_1 m,$$

from where $c_{01} = 2.5 \times 4 = 10$ and the utility of an initial old person is $\sqrt{10} \approx 3.16$.

(f) **(5 points)** We never used the actual size of $m$ before part 5. So therefore now:

$$v_t = \frac{v_t m_t}{m_t} = \frac{10}{8} = 1.25,$$

or, intuitively, now a single unit of money can only buy half of what it could before. This is called “neutrality of money” effect, scaling up the money stock only raised all prices proportionately (you should think of falling $v_t$ as of inflation since $p_t = \frac{1}{v_t}$). The initial old will attain the same level of utility, however, because the real purchasing power of money didn’t change: they now have twice as many units of currency and each unit can buy half as much stuff.

**Problem 2 (20 points)**

This problem mimics Exercise 1.2 from the CF textbook. Consider two overlapping generations economies, $A$ and $B$. Both economies have the same constant population $N$, and the supply of fiat money in each economy is equal to $M$. In both economies every young person is endowed with $y > 0$ units of consumption good. Every old person has nothing (except for the initial old, each such person holds $\frac{M}{N}$ units of fiat money).

The only difference between these economies is with regard to preferences that people have. Other things being equal, individuals in economy $A$ have preferences that lean towards first period consumption; individual preferences in economy $B$ lean towards second period consumption. For concreteness, you may want to think that people in economy $A$ have utility function

$$u^A (c_{1,t}^A, c_{2,t+1}^A) = 2 \log c_{1,t}^A + \log c_{2,t+1}^A,$$

and people in economy $B$ have utility function

$$u^B (c_{1,t}^B, c_{2,t+1}^B) = \log c_{1,t}^B + 2 \log c_{2,t+1}^B.$$

We will focus on the stationary monetary equilibrium in each of these economies. You will not need the information on individual utility functions in order to answer this question.

Do the following:

(a) Will there be a difference in rates of return on fiat money in these two economies? If yes, than which economy do you expect to have a higher rate of return on money? Give an intuitive interpretation of your answer and provide a concise formal justification. You do not have to compute the actual rates of return to answer this question.

(b) Will there be a difference in the value of money in these two economies? If yes, than which economy do you expect to have a higher value of money? Give an intuitive interpretation of your answer and provide a concise formal justification. Again, you do not have to compute the actual rates of return to answer this question.
Solution

(a) (10 points) The real rate of return on fiat money is
\[ \frac{v_{t+1}}{v_t} = \frac{N_t (y - c_t)}{M_t} = 1, \]
as long as \( M_t = M \) and \( N_t = N \) for all \( t = 1, 2, 3, \ldots \) So both economies will have the same rate of return on fiat money. Intuitively, these rates would be different if economies \( A \) and \( B \) evolved differently over time. For example, if population \( A \) was growing, and it stayed constant in \( B \), then return on money would have been different.

(b) (10 points) In a stationary equilibrium, it is true that:
\[ v_t^A = \frac{N (y - c_t^A)}{M} \]
\[ v_t^B = \frac{N (y - c_t^B)}{M} \]
and since by assumption, population in economy \( A \) is more geared towards consuming when young, we will have \( c_t^A > c_t^B \), and hence
\[ v_t^A < v_t^B \]
for all \( t = 1, 2, 3, \ldots \) Intuitively, the demand for money will be larger in economy \( B \) than in economy \( A \). This is because individuals in economy \( B \) want to hold relatively more money to finance their higher second-period consumption. Since all else is equal between the two economies (importantly, the supply of money and population), money will have a higher value in economy \( B \) than in economy \( A \).

Problem 3 (50 points)
Consider an overlapping generations model in which consumers live for two periods. The number of people born in each generation is constant, \( N_t = N_{t+1} = N \). In each period, young consumers are endowed with \( y_1 = 30 \) and old consumers are endowed with \( y_2 = 0 \) units of the single consumption good. Each member of the generations born in period 1 and later have the following utility function:
\[ u(c_{1,t}, c_{2,t+1}) = \log [c_{1,t}] + \beta \log [c_{2,t+1}], \]
with \( \beta = 0.5 \).
Members of the initial old generation only live for one period and have utility \( u(c_{0,1}) = \log [c_{0,1}] \). They have no endowment.

(a) Define a feasible consumption allocation for this economy. Illustrate the set of feasible allocations on a graph.

(b) Define a Pareto efficient stationary allocation for this economy.
(c) Solve for the Pareto efficient stationary allocation.

(d) Define a **competitive equilibrium without money** for this economy. Will there be any trades between individuals in the economy? Explain.

(e) Solve for the consumption allocation in the competitive equilibrium without money. Is the allocation the same as in (c)? Which consumption allocation does each generation prefer (i.e. compare the competitive equilibrium allocation and the Pareto optimal allocation)?

(f) Now suppose that each member of the initial old generation is endowed with $m_0$ units of fiat money. (Therefore, the stock of fiat money in the economy $M = N \cdot m_0$.) Define a **competitive equilibrium with money** for this economy.

(g) Solve for the consumption allocation and the demand for real balances $(v_t, m_t)$ in the competitive equilibrium with money as a function of the **rate of return of fiat money** $(\frac{v_{t+1}}{v_t})$.

(h) Find the rate of return of fiat money and the consumption allocation in the stationary competitive equilibrium with money.

(i) Suppose that $m_0 = 10$, that is, each initial old person is endowed with 10 units of money. Find the demand for fiat money $m_t$ and the value of fiat money $v_t$ in every period.

**Solution**

(a) **(5 points)** An allocation is a pair of consumption quantities $(c_{1,t}, c_{2,t+1})$ for every guy born at time $t \geq 1$, plus $c_{0,1}$ for the initial old. A feasible allocation is an allocation that has the following property: for all $t \geq 1$:

$$N_t c_{1,t} + N_{t-1} c_{2,t} \leq N_t y$$

Any version of the above equation that exploits the fact that $N_t = N$ for all $t$ or that $y = 30$ would also be a correct answer. Presumably everyone should be able to draw this on a diagram.

(b) **(5 points)** A Pareto efficient allocation is a feasible allocation with the property that there does not exist any other feasible allocation that makes at least one agent in the economy better-off without making anyone else worse-off.

Alternatively, a Pareto efficient allocation is a solution to the following sequence of problems for all $t \geq 1$:

$$\max_{c_{1,t}, c_{2,t}} \log c_{1,t} + \beta \log c_{2,t}$$

s.t. (1)

with the property that $c_{2,1} = c_{0,1}$.

(c) **(5 points)** We solve the problem (2)-(1) above. First we simplify the constraint to $c_{1,t} + c_{2,t} = y$. Then, we form the Lagrange function (for sure there are other ways to solve the problem):

$$L = \log c_{1,t} + \beta \log c_{2,t} + \lambda (y - c_{1,t} - c_{2,t})$$

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and take the F.O.C.s:

\[
\frac{1}{c_{1,t}} = \lambda \quad \quad (5)
\]
\[
\frac{\beta}{c_{2,t}} = \lambda \quad \quad (6)
\]
\[
c_{1,t} + c_{2,t} = y \quad \quad (7)
\]

Now equate the left hand sides of the two first conditions to get:

\[
e_{2,t} = \beta c_{1,t} \quad \quad (8)
\]
and plug this condition into the third F.O.C. to get:

\[
c_{1,t} + \beta c_{1,t} = y \quad \quad (9)
\]
\[
c_{PO}^{1,t} = \frac{1}{1+\beta} y \quad \quad (10)
\]
and hence:

\[
c_{PO}^{2,t} = \frac{\beta}{1+\beta} y \quad \quad (11)
\]

We now substitute the numbers from the problem to get \( c_{PO}^{1,t} = 20 \) and \( c_{PO}^{2,t} = 10 \). This would be our answer. Notice that this allocation is stationary – it does not depend on \( t \).

(d) (5 points) A competitive equilibrium without money for this economy would be an allocation \((\tilde{c}_{1,t}, \tilde{c}_{2,t+1})\) for every person born at time \( t \geq 1 \) and \( \tilde{c}_{0,1} \) for the initial old such that:

1. For every \( t \geq 1 \), every person born at \( t \) chooses \((\tilde{c}_{1,t}, \tilde{c}_{2,t+1})\) to solve:

\[
\max_{c_{1,t},c_{2,t+1}} \log c_{1,t} + \beta \log c_{2,t+1} \quad \quad (12)
\]
\[
\text{s.t.} \quad \begin{cases} 
  c_{1,t} \leq y \\
  c_{2,t+1} \leq 0
\end{cases} \quad \quad (13)
\]

2. Initial old choose \( \tilde{c}_{0,1} \) to solve:

\[
\max_{c_{0,1}} \log c_{0,1} \quad \text{s.t.} \quad c_{0,1} \leq 0 \quad \quad (14)
\]

3. All markets clear – here there is only one market for consumption, and the condition is given by (1), which has to hold for all \( t \geq 1 \).

A competitive equilibrium without money must involve no trade. Consumers are not willing to exchange because the young cannot trade with the old. There is absence of double coincidence of wants. The old consumers want what the young have (consumption now) but the old don’t have what the young want (consumption tomorrow), because the old will not be alive tomorrow.
(e) (5 points) Solving for the competitive equilibrium without money is very easy – the constraints for each problem pin down the optimal quantities uniquely. So \((\tilde{c}_{1,t}, \tilde{c}_{2,t+1}) = (y, 0) = (30, 0)\) and \(\tilde{c}_{0,1} = 0\). Obviously the market clearing constraint (1) holds for this allocation.

The allocation is NOT the same as in (c), where \(c_{1,t}^{PO}\) and \(c_{2,t}^{PO}\) were both positive. Everybody in the economy prefers the PO allocation:

1. The initial old have: \(\log c_{0,1}^{PO} = \log 10 > \log 0 = \log \tilde{c}_{0,1}\)
2. Every other guy has: \(\log c_{1,t}^{PO} + \beta \log c_{2,t+1}^{PO} = 20 + \frac{1}{2} \log 10 > \log 30 + \frac{1}{2} \log 0 = \log \tilde{c}_{1,t} + \beta \log \tilde{c}_{2,t+1}\).

(f) (5 points) A competitive equilibrium with money for this economy would be an allocation \((c_1^*, t_{1}, c_2^*, t_{1}+1)\) for every person born at time \(t \geq 1\), \(c_0^*, 1\) for the initial old and prices of money \(v_t^*\) for all \(t \geq 1\) such that:

1. For every \(t \geq 1\), every person born at \(t\), taking \(v_t^*\) and \(v_{t+1}^*\) as given, chooses \((c_1^*, t_{1}, c_2^*, t_{1}+1)\) to solve:

\[
\max_{c_1^*, t_{1}+1, c_2^*, t_{1}+1} \log c_1 + \beta \log c_2 + \lambda (y-c_1) \tag{15}
\]

\[
s.t. \quad \begin{cases} 
  c_1 + v_t^* m_t \leq y \\
  c_{2,t+1} \leq v_{t+1}^* m_t 
\end{cases} \tag{16}
\]

2. Initial old choose \(c_0^*, 1\) to solve, given \(v_t^*\):

\[
\max_{c_0^*, 1} \log c_0 \quad \text{s.t.} \quad c_0 \leq v_1^* m_0 \tag{17}
\]

3. All markets clear for all \(t \geq 1\):

\[
N_t c_{1,t} + N_{t-1} c_{2,t} \leq N_t y \tag{18}
\]

\[
N_t m_t \leq N m_0 \tag{19}
\]

where the first market-clearing condition above refers to the consumption market, and the second refers to the money market.

(g) (10 points) We solve the problem of a representative guy by creating a lifetime budget constraint:

\[
\begin{cases} 
  c_1 + v_t^* m_t \leq y \\
  c_{2,t+1} \leq v_{t+1}^* m_t 
\end{cases} \iff \begin{cases} 
  c_1 + v_t^* m_t = y \\
  c_{2,t+1} = v_{t+1}^* m_t 
\end{cases} \tag{20}
\]

\[
m_t = \frac{c_{2,t+1}}{v_{t+1}^*} \tag{21}
\]

\[
c_1 + \frac{v_t^*}{v_{t+1}^*} c_{2,t+1} = y \tag{22}
\]

forming the Lagrange function:

\[
L = \log c_1 + \beta \log c_{2,t+1} + \lambda \left( y - c_1 - \frac{v_t^*}{v_{t+1}^*} c_{2,t+1} \right) \tag{23}
\]
and taking F.O.C.s:
\[
\begin{align*}
\frac{1}{c_{1,t}} &= \lambda \\
\beta \frac{c_{2,t+1}}{v_{t+1}^*} &= \lambda \frac{v_{t+1}^*}{v_{t+1}^*} \\
y &= c_{1,t} + \frac{v_{t+1}^*}{v_{t+1}^*} c_{2,t+1}
\end{align*}
\]  

The usual technique can be applied here: divide the first condition by the second and rearrange to get:
\[
c_{2,t+1} = \beta \frac{v_{t+1}^*}{v_{t}^*} c_{1,t}
\]  

and substitute into the last condition:
\[
y = c_{1,t} + \frac{v_{t}^*}{v_{t}^*} \left[ \beta \frac{v_{t+1}^*}{v_{t}^*} c_{1,t} \right]
\]
\[
c_{1,t}^* = \frac{1}{1+\beta} y
\]
\[
c_{2,t+1}^* = \frac{\beta}{1+\beta} \frac{v_{t+1}^*}{v_{t}^*} y
\]

So \( c_{1,t}^* = 20 \) and \( c_{2,t+1}^* = 10 \frac{v_{t+1}^*}{v_{t}^*} \). We also want to find the demand for real money \( v_t^* m_t \). For this, we first obtain \( m_t \):
\[
m_t = \frac{c_{2,t+1}^*}{v_{t+1}^*} = \frac{1}{v_{t+1}^*} \left( \frac{\beta}{1+\beta} \frac{v_{t+1}^*}{v_{t}^*} y \right)
\]
and hence \( v_t^* m_t = \frac{\beta}{1+\beta} y = 10 \).

(h) (5 points) Now assume stationarity, so \( c_{1,t}^* = c_{1}^* \) for all \( t \) and \( c_{2,t+1}^* = c_{2}^* \) for all \( t \). To pin down the rate of return on money, \( \frac{v_{t+1}^*}{v_{t}^*} \), we use the money market clearing condition:
\[
N_t m_t = N m_0
\]
and cancel \( N_t \) with \( N \) since it is constant by assumption:
\[
m_t = m_0
\]
\[
v_t^* m_t = v_t^* m_0
\]
Now we use the budget constraint of a young guy: \( v_t^* m_t = y - c_t^* \):
\[
y - c_t^* = v_t^* m_0
\]
\[
y - \frac{1}{1+\beta} y = v_t^* m_0
\]
\[
\frac{\beta}{1+\beta} m_0 = v_t^*
\]
Hence \( v_t^* = \frac{10}{m_0} \) for all \( t \). Therefore \( \frac{v_{t+1}^*}{v_t^*} = 1 \) and hence we know the consumption allocation: \( c_1^* = 20 \), \( c_2^* = 10 \). Notice that we have \( (c_1^*, c_2^*) = (c_1^{PO}, c_2^{PO}) \).

(i) (5 points) Now, if \( m_0 = 10 \), we have \( v_t^* = \frac{10}{10} = 1 \) and \( m_t = \frac{\beta}{1+\beta} y = \frac{1}{1+0.5} 30 = 10 \).